# The Aesthetic Space Curve

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## Abstract

The curve is the most basic design element to determine shapes and silhouettes of industrial products and works for shape designers and it is inevitable for them to make it aesthetic and attractive to improve the total quality of the shape design. If we can find equations of the aesthetic curves, it is expected that the quality of the curve design improves drastically because we can use them as standards to generate, evaluate, and deform the curves. The authors have proposed the general equations of aesthetic curves as such a standard.

However the aesthetic curves expressed by the general equations are limited to planar curves. Hence in this paper, at first we show the necessary and sufficient condition for a given curve to have the self-affinity and then extend the aesthetic curves into 3-dimensional space.

**Keywords:** aesthetic curve, spacial aesthetic curve, self-affinity

## 1 Introduction

"Aestehtic curves" were proposed by Harada et al. [1] as such curves whose logarithmic distribution diagram of curvature is approximated by a straight line. Miura [2, 3] derived analytical solutions of the curves whose LDDC are strictly given by a straight line and proposed these lines as general equations of aesthetic curves. Furthermore, Yoshida and Saito [4] analyzed the properties of the curves expressed by the general equations and developed a new method to interactively generate a curve by specifying two end points and the tangent vectors there with three control points as well as $\alpha$: the slope of the straight line of the LDDC. In this research, we call the curves expressed by the general equations of aesthetic curves the aesthetic curves.

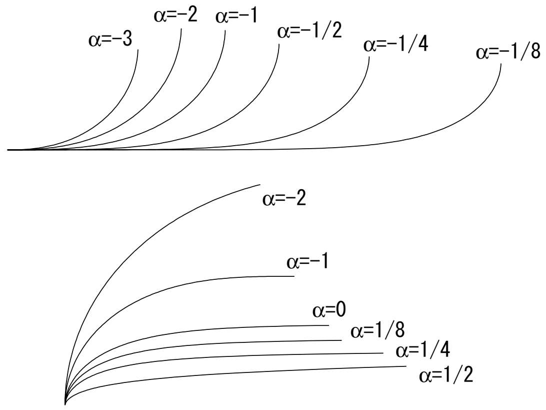


Figure 1: Aesthetic plane curves with various $\alpha(\gamma)$ values

The aesthetic curves include the logarithmic (equiangular) curve ($\alpha = 1$), the clothoid curve ($\alpha = -1$) and the involute curve ($\alpha = 2$). It is possible to generate and deform the aesthetic curve even if they are expressed by integral forms using their unit tangent vectors as integrands ($\alpha \neq 1, 2$) and they are expected to be used in practical applications. However, they are planar and it is not possible to represent spacial curves. Therefore, in this paper, at first we will show the necessary and sufficient condition for planar curves to have self-affinity and extend the aesthetic curves into 3-dimensional space with guaranteeing self-affinity. We call the derived curve the aesthetic space curve.

# 2 Aesthetic plane curve

In this reseach, we define the aesthetic curve as the curve whose logarithmic distribution diagram of curvature is stricly expressed by a straight line.

## 2.1 General equations of aesthetic curves

For a given aesthetic curve we assume that the arc length of the curve is given by $s$ and the radius of curvature $\rho$, the horizontal axis of the LDDC measures $\log \rho$ and the vertivcal axis $\log(ds/d(\log \rho)) = \log(\rho\, ds/d\rho)$. Since its LDDC is given by a straight line, there exists some constant $\alpha$ and the folloing equation is satisfied:

$$\log(\rho \frac{ds}{d\rho}) = \alpha \log \rho + C \tag{1}$$

where $C$ is a constant. We call this the fundamental equation of aesthetic curves. . Rewrite Eq.(1) and it becomes

$$\frac{1}{\rho^{\alpha-1}} \frac{ds}{d\rho} = e^C = C_0 \tag{2}$$

Hence there is some constant $c_0$ such that

$$\rho^{\alpha-1} \frac{d\rho}{ds} = c_0 \tag{3}$$

Figure 1 shows several aesthetic plance curves with various $\alpha$ values.

## 2.2 Self-affinity of the plane curves

We define self-affinity of the plane curve as follows [3]. **Self-affinity of the plane curve**: For a curve generated by removing arbitrary head portion of the original curve, by scaling it with different factors in its tangent and normal directions on every point on the curve, if the original curve is obtained, then the curve has self-affinity.

If a given plane curve satisfys Eq.(3), the curve has self-similarity of this definition [2].

## 2.3 A necessary and sufficient condition for self-affinity

For a given curve $\boldsymbol{C}(s)$ parametrerized by the arc length parameter $s \geq 0$, we assume the derivative of its curvature, hence that of its radius of curvature as well are continuous. I other words, we assume the curve has $C^3$ continuity. In addition, the radius of curvature $\rho(s)$ is assumed not to be equal to 0.

By scaling the curve with different factors in the tangent and normal directions (affine transformation of the plane curve [3]), we think about how to make the scaled curve become congruent with the original curve. We therefore reparameterize the given curve $\boldsymbol{C}(s)$ using a new parameter $t = as + b$ where $a$ and $b$ are positive constants as shown in Fig.2.3. To scale the curve uniformly in the tangent direction is equivalent to relate a point $\boldsymbol{C}(t_0 = as_0 + b)$ to another point $\boldsymbol{C}(s_0)$ as shown in Fig.2.3. In this relationship the scaling factor in the tangent direction $f_t$ is given by $1/a$.

Although $a$ and $b$ are constants, they are related to the scaling facotors in the tangent and normal directions $f_t$ and $f_n$ and they depends on the shape of the curve. Hence we can not specify them independently.

The start point of the curve $\boldsymbol{C}(t)$ is given by $\boldsymbol{C}(b)$ that is a point when $s = 0$. Hence $\boldsymbol{C}(t)$ is a curve without the head porition of the origianl curve $\boldsymbol{C}(s)$.

The condition can be described for a curve to have self-affinity by the following. **Condition for a plane curve to have self-affinity**: For an aribitray constant $b > 0$, some $a > 0$ is determined. With these $a$ amd $b$, for any $s \geq 0$ the following equation is satisfied.

$$\frac{\rho(s)}{\rho(as+b)} = f_n \tag{4}$$

where $f_n$ is a constant dependent on and determined by $b$ and it is a scaling factor in the normal

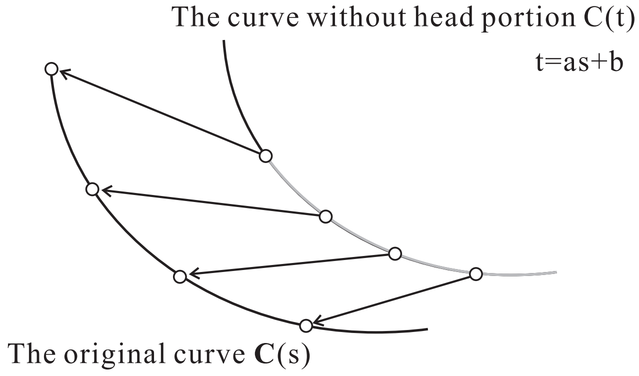


Figure 2: Correspondence of the reparameterized and original curves

direction. $f_n$ is given by substituing $s = 0$ in the above equation as follows:

$$f_n = \frac{\rho(0)}{\rho(b)} \tag{5}$$

### 2.3.1 In case of $f_n = 1$

To make the following arguments simpler, at first we discuss the case where $f_n = 1$. From Eq.(4),

$$\rho(s) = \rho(as + b) \tag{6}$$

By the lemma in the appendix, $\rho(s)$ turns out to be constant and the curve is given by an arc or a straight line ($\rho(s) = \infty$).

In what follows, $f_n \neq 1$ is assumed. By rewriting Eq.(4),

$$\rho(s) - f_n\, \rho(as + b) = 0 \tag{7}$$

Since the radius of curvature $\rho(s)$ is differentiable,

$$\begin{aligned} \frac{d\rho(s)}{ds} \quad &- \quad a\, f_n\, \frac{d\rho(t)}{dt}\bigg|_{t=as+b} \\ &= \frac{d\rho(s)}{ds} - \frac{f_n}{f_t}\frac{d\rho(t)}{dt}\bigg|_{t=as+b} = 0 \end{aligned} \tag{8}$$

By substituting 0 for $s$ and rewriting the above equation,

$$f_t = f_n \frac{\frac{d\rho(b)}{dt}}{\frac{d\rho(0)}{ds}} \tag{9}$$

Hence as Eq.(5) is satisfied, both $f_n$ and $f_t$ are determined uniquely by the values of the radius of curvature and its derivative at the start point of the curve[1].

[1] From $a = 1/f_t$, $a$ is also uniquely determined by $b$.

### 2.3.2 In case of $f_n/f_t = 1$

First, for some $b > 0$, if $f_n/f_t = 1$, by the similar arguments to subsection 2.3.4, for an arbitrary $b$, $f_n/f_t$ is equal to 1. Then

$$\frac{d\rho(s)}{ds} = \frac{d\rho(t)}{dt}\bigg|_{t=as+b} \tag{10}$$

From this equation and the lemma,

$$\frac{d\rho(s)}{ds} = c_0 \tag{11}$$

$c_0$ is constant and by integrating the above equajtion,

$$\rho(s) = c_0\, s + c_1 \tag{12}$$

where $c_1$ is a constant of integration. Eq.(12) represents the relationship between the radius of curvature and the arc length of the logarithmic spiral and the curva has a special self-affinity, i.e., self-similarity where $f_t$ is equal to $f_n$.

### 2.3.3 In case of $f_n/f_t \neq 1$

Next, if $f_n/f_t \neq 1$, since $f_n \neq 1$, there is some $\alpha$ such that

$$\frac{f_n}{f_t} = f_n^{1-\alpha} \tag{13}$$

Then

$$\begin{aligned} \frac{d\rho(s)}{ds} &= f_n^{1-\alpha}\frac{d\rho(t)}{dt}\bigg|_{t=as+b} \\ \frac{d\rho(s)}{ds} &= \left\{\frac{\rho(s)}{\rho(as+b)}\right\}^{1-\alpha}\frac{d\rho(t)}{dt}\bigg|_{t=as+b} \end{aligned} \tag{14}$$

Hence

$$\rho(s)^{\alpha-1}\frac{d\rho(s)}{ds} = \rho(as+b)^{\alpha-1}\frac{d\rho(t)}{dt}\bigg|_{t=as+b} \tag{15}$$

Threfore, if $\alpha$ is independent from $b$, from the lemma, we obtain the following equation equivalent to Eq.(3).

$$\rho(s)^{\alpha-1}\frac{d\rho(s)}{ds} = c_0 \tag{16}$$

where $c_0$ is a constant. By integrating the above equation, the first and second general equations are derived [2].

### 2.3.4 Independence of $\alpha$ from $b$

In this subsection, we will prove that $\alpha$ is independent from $b$. Here we think about the case where $b$ is small enough and let it to be $\Delta b > 0$. Let $a$ to be $1 + \Delta a$ or $1 - \Delta a$, $(\Delta a > 0)$ depending on and uniquely determined by $\Delta b$. We relax the condition that $b$ is positive and think about the case where $b = 0$ and let $\Delta b$ is equal to 0. Then Eq.(4) relates itself. Hence $a = 1$, or $\Delta a = 0$. Then $f_n = 1$. For the curve without the portion corresponding the domain $0 \leq s < \Delta b$, Eq.(4) is satisfied and from Eq.(13), there exists $\alpha$ such that

$$\frac{\rho(s)}{\rho((1 \pm \Delta a)s + \Delta b)} = f_n = \left\{\frac{f_n}{f_t}\right\}^{1-\alpha} \tag{17}$$

$a$ is a continuous function of $b$ and we can make the value of $\Delta a$ smaller without limit if we make $\Delta b$ smaller.

In Eq.(4), by substituting $(1 \pm \Delta a)s + \Delta b$ for $s$ repeatedly,

$$\begin{aligned} f_n &= \frac{\rho(s)}{\rho((1 \pm \Delta a)s + \Delta b)} \\ f_n &= \frac{\rho((1 \pm \Delta a)s + \Delta b)}{\rho((1 \pm \Delta a)^2 s + \Delta b((1 \pm \Delta a) + 1))} \\ &\cdots \\ f_n &= \frac{\rho((1 \pm \Delta a)^{m-1} s + \cdots + 1))}{\rho((1 \pm \Delta a)^m s + \cdots + 1))} \end{aligned}$$

where $\pm$ is appropriately selected for the given curve to satisfy $\Delta a > 0$. From these equations,

$$\frac{\rho(s)}{\rho((1 \pm \Delta a)^m s + \Delta b((1 \pm \Delta a)^{m-1} + \cdots + 1))} = f_n^m$$

Hence the scaling factor in the tangent direction for $b = \Delta b((1 \pm \Delta a)^{m-1} + \cdots + 1)$ is equal to $1/(1 \pm \Delta a)^m = f_t^m$ and

$$f_n^m = \left\{\frac{f_n^m}{f_t^m}\right\}^{1-\alpha} \tag{18}$$

Therefore $\alpha$ is equal to that for $\Delta b$.

We will prove that $\alpha$ is constant by contradiction. From Eq.(13), $\alpha$ is expressed by a continuous function of $b$ and $\alpha = \alpha(b)$. For some $b_0 > \Delta b > 0$, $\alpha_0 = \alpha(b_0)$ and we assume that $\alpha_0$ is different from $\alpha = \alpha(\Delta b)$. For a positive small $\epsilon$, we furthermore assume that

$$|\alpha_0 - \alpha| > 2\epsilon \tag{19}$$

Since $\alpha(b)$ is a continuous function, there exists some $\delta$ such that for any $b > 0$ satisfying $|b_0 - b| < \delta$

$$|\alpha(b_0) - \alpha(b)| < \epsilon \tag{20}$$

As $\Delta a$ is small, $1 \pm \Delta a > 0$ and $\Delta b((1 \pm \Delta a)^{m-1} + \cdots + 1))$ increases monotonously from $\Delta b$ and can becomes larger than any value by increasing $m$. Hence there exists $m$ such that

$$\begin{aligned} b_l &= \Delta b((1 \pm \Delta a)^{m-1} + \cdots + 1)) < b_0 \\ &< b_u = \Delta b((1 \pm \Delta a)^m + \cdots + 1)) \end{aligned} \tag{21}$$

Since $b_u - b_l = \Delta b(1 \pm \Delta a)^m$, if

$$\Delta b(1 \pm \Delta a)^m < 2\delta \tag{22}$$

we get $|b_0 - b_l| < \delta$ or $|b_0 - b_u| < \delta$. Eq.(22) can be rewritten into $1 \pm \Delta a < (2\delta/\Delta b)^{\frac{1}{m}}$ and $\Delta a$ becomes smaller if we make $\Delta b$ smaller and there exists $\Delta b$ satisfying this equation. Hence Eq.(20) is satisfied and contradicts (19). Therefore $\alpha$ is constant for any $b$.

To sum up the results of the above discussions, a necessary and sufficient condition for the plane cuve to have self-affinity is that for some constant $\alpha$, Eq.(16) is satisfied. When $\alpha = 1$, Eq.(16) becomes Eq.(11) and it icludes the case of self-similarity.

## 2.4 Self-affinity ratio

$\alpha$ is a slope of the graph of logarithmic distribution diagram of curvature and as discussed in the previous section it has relationship with the scaling factors in the tangent and normal directions $f_t$, $f_n$. It characterizes the curve. From Eq.(13), let $\gamma$ to be the reciprocal of $\alpha$. Then

$$\gamma = \frac{1}{\alpha} = \frac{\log f_n}{\log f_t} \tag{23}$$

This means $f_n = f_t^\gamma$.

For fractals who has self-similarity, as a measure to represent their dimensions similarity dimension is defined as follows [5]. When the whole figure consists of similar figures of number $1/b$ scaled by $1/a$, $b = a^D$ and similarity dimension is given by

$$D = \frac{\log b}{\log a} \tag{24}$$

Eq.(23) is similar to the above definition and Eq.(23) can be interpreted that it is necessary to have curves of number $f_n$ to fill up the space in the normal direction if we scale the curve by $1/f_t$. $\gamma$ can be interpreted as a dimension and we call it self-affinity ratio.

# 3 Extension into 3-dimensional space

The aesthetic curve proposed so far is a plane curve and we extend it into 3-dimensional space by using the Frenet-Serret formula (for example, see [6]).

## 3.1 The Frenet-Serret formula

For a space curve $\boldsymbol{C}(s)$ parameterized by $s$, let its unit tangent vector to be $\boldsymbol{t}$, unit principal normal vector $\boldsymbol{n}$, and unit binormal vector$\boldsymbol{b}$. These vectors are related by the Frenet-Serret formula as follows:

$$\begin{aligned} \frac{d\boldsymbol{C}(s)}{ds} &= \boldsymbol{t}, \quad \frac{d\boldsymbol{t}}{ds} = \kappa\boldsymbol{n}, \\ \frac{d\boldsymbol{n}}{ds} &= -\kappa\boldsymbol{t} + \tau\boldsymbol{b}, \quad \frac{d\boldsymbol{b}}{ds} = -\tau\boldsymbol{n} \end{aligned} \quad (25)$$

where $\kappa$ and $\tau$ are the curvature and torsion, respectively.

The plane curve has a constant binormal vector and its torsion remains 0. But we have to consider its change for the space curve. Hence first, we define self-affinity of the space curve and next we define the aesthetic space curve as the curve who has self-affinity.

Similar to self-affinity of the plane curve, we define self-affinity of the space curve as follows. **Self-affinity of the space curve**: For a curve generated by removing arbitrary head portion of the original curve, by scaling it with different factors in its tangent, principal normal and binormal directions on every point on the curve, if the original curve is obtained, then the curve has self-affinity.

Since the curvature and torsion, or their reciprocals: the radius of curvature and radius of torsion can be independently specified, With respect to the radius of torsio $\mu = 1/\tau$, we assume that an equation similar to Eq.(1) is satisfied as follows:

$$\log(\mu\frac{ds}{d\mu}) = \beta\log\mu + C' \quad (26)$$

where $\beta$ is a constant. Then

$$\mu^{\beta-1}\frac{d\mu}{ds} = c_1 \quad (27)$$

Arguments similar to those where that subsection 2.3 has showed that a sufficient and necessary condition to have self-affinity of the plane curve is expressed by Eq.(3) can show that a sufficient and necessary condition to have self-affinity of the space curve is expressed by Eqs.(3) and (27).

The Frenet-Serret formula can be considered to be simultanious differential equations and an example calculated by their numerial integration is shown in Fig.3. The left and right figures shows the same five curves from different viewpoints and the curve drawn at the bottoms is identical to a logarithmic spiral whose torsion is always 0 and radius of curvature is given by a linear function of the arc length. The other curves have the same start point and radius of curvature as the logarithmic spiral and their torsion is given by a linear function of the arc length with $\beta = 1$. The upper curves have smaller coefficient of the linear function for the arc length (larger torsion). For each curve, at the start and end points, and two points on the curve, we draw the tangent, principal normal and binormal vectors of the moving frame (Frenet frame) as short slim cyliders.

# 4 Conclusions

In this research, we have derived sufficient and necessary conditions for the plane curve and the space curve to have self-affinity and extended the aesthetic plane curve into 3-dimensional space with self-affinity based on the Frenet-Serret formula and derived the aesthetic space curve. For the aesthetic space curve, the radius of torsion, i.e., the reciprocal of torsion to the power of some constant is given by a linear function of the arc length similar to the radius of curvature. We guarantee self-affinity of the aesthetic space curve.

For future work, we are planning an automatic classification of curves: 1) determine the rhythm to be simple(monotonic) or complex(consisting of plural rhythms), 2) calculate the slope of the line approximating the LDDC graph. We think there are a lot of possibilities to use the general aesthetic equations to many applications in the fields

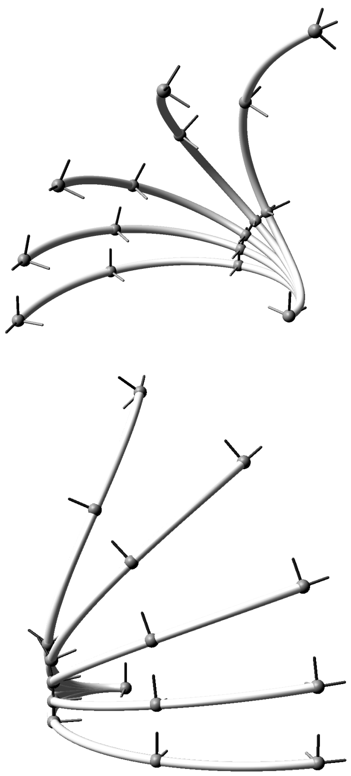

Figure 3: Examples of the aesthetic space curve

of computer aided geometric design. For example, we may be able to apply the equations to deform curves to change their impressions, say, from sharp to stable. Another example is smoothing for reverse engineering. Even if only noisy data of curves are available, we may be able to use the equations as kinds of rulers to smooth out the data and yield aesthetically high quality curves. We will develop a CAD system using the aesthetic plane and space curves.

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# Appendix

## A Lemma

Here a function $f(s)$ patameterized by $s$ is given. For an arbitrary constant $b > 0$, $a > 0$ is determined by $b$. With these $a$ and $b$, if for any $s \geq 0$ the following equation is assumed to be satisfied.

$$f(a\,s + b) = f(s) \tag{28}$$

Then the function $f(s)$ is always constant.

**Proof**: Assume the function $f(s)$ is not constant. Then there exists such $s_0 > 0$ that

$$f(s_0) \neq f(0) \tag{29}$$

If $b = s_0$, for some $a_0 > 0$,

$$f(a_0\,s + s_0) = f(s) \tag{30}$$

By substituting 0 for $s$ に 0, $f(s_0) = f(0)$ is obtained and that contradicts Eq.(29). Therefore, $f(s)$ is constant[2].

[2] The lemma means that for an arbitrary $b > 0$, $a = a(b) > 0$, when the given function is scaled by $a$ about the origin and is translated by $b$, if the function is congruent with the original function, then the funtion is constant.